

THEORY OF $\pi\pi$ SCATTERING

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I describe the current status of the theory of $\pi\pi$ scattering, reviewing in particular recent work on the numerical solution of Roy equations and on the matching between these and the chiral representation. I discuss numerical results on the scattering lengths and other threshold parameters.

1 Introduction

The study of $\pi\pi$ scattering is a classical subject in the field of strong interactions: it is a scattering process that may occur at very low energy, far away from the resonance region, and it only involves the quasi-Goldstone bosons of the chiral symmetry of strong interactions. If we stay at low energy, we could expect its dynamics to be strongly constrained by chiral symmetry: as Weinberg showed ¹, in the chiral limit, the scattering amplitude has to vanish when the momenta of the pions tend to zero. In the real world quarks are not massless, and consequently also the would-be Goldstone bosons acquire a small mass. The chiral argument given above implies that the amplitude at threshold must be proportional to the square of the mass of the pions ¹:

$$a_0^0 = \frac{7M_\pi^2}{32\pi F_\pi^2}, \quad a_0^2 = -\frac{M_\pi^2}{16\pi F_\pi^2}, \quad (1)$$

where a_ℓ^I stands for the scattering length in the isospin I channel with angular momentum ℓ .

The formulae above are valid only at leading order in a series expansion in powers of the quark masses: the next-to-leading order corrections have been calculated by Gasser and Leutwyler ², and even the next-to-next-to-leading order corrections are now known ³. The latter had been calculated already before the previous edition of the “Chiral Dynamics” series of Workshops, and the interested reader may find a report of the situation at that time in the contribution by Gerhard Ecker to the previous Workshop Proceedings ⁴. The main difference between that report and the present one is the fact that at that time the numerical analysis had not been completed: in order to illustrate the numerical size of the two-loop contributions, Ecker had to present three different numerical values of the S -wave scattering lengths, depending on the

input used for the low energy constants (see table 1 in Ref. ⁴). The numerical analysis has been recently completed ⁶, and has yielded the values:

$$a_0^0 = 0.222 \pm 0.005, \quad a_0^2 = -0.0445 \pm 0.001. \quad (2)$$

In what follows I will describe what are the experimental ingredients and the theoretical tools that have lead to this result.

2 Numerical solutions of Roy equations

In 1971 Roy ⁷ derived a set of dispersion relations for the partial wave amplitudes of $\pi\pi$ scattering, that fully respected crossing symmetry, and that involved only two subtraction constants. The crucial observation that lead to this result was that if one writes a fixed- t dispersion relation, crossing symmetry constrains the form of the t dependent subtraction constants in such a way that they can be expressed in terms of the two S -wave scattering lengths and other dispersive integrals. The equations have the following form:

$$t_\ell^I(s) = k_\ell^I(s) + \sum_{I'=0}^2 \sum_{\ell'=0}^{\infty} \int_{4M_\pi^2}^{\infty} ds' K_{\ell\ell'}^{II'}(s, s') \text{Im } t_{\ell'}^{I'}(s'), \quad (3)$$

where I and ℓ denote isospin and angular momentum, respectively and $k_\ell^I(s)$ is the partial wave projection of the subtraction term, present only in S - and P -waves,

$$k_\ell^I(s) = a_0^I \delta_\ell^0 + \frac{s - 4M_\pi^2}{4M_\pi^2} (2a_0^0 - 5a_0^2) \left(\frac{1}{3} \delta_0^I \delta_\ell^0 + \frac{1}{18} \delta_1^I \delta_\ell^1 - \frac{1}{6} \delta_2^I \delta_\ell^0 \right). \quad (4)$$

The kernels $K_{\ell\ell'}^{II'}(s, s')$ are explicitly known functions. They contain a diagonal, singular Cauchy kernel that generates the right hand cut in the partial wave amplitudes, as well as a logarithmically singular piece that accounts for the left hand cut. Soon after these equations became available in the literature, various groups started to treat them numerically ⁸. At that time the goal was to derive from the available data constraints on the scattering lengths: all groups agreed on the result that only data sufficiently close to threshold could provide significant bounds on the scattering lengths. With the only exception of data on K_{e4} decays, all other data were at too high an energy, and therefore left the S -wave scattering lengths practically unconstrained.

The only high-statistics experiment on K_{e4} decays provided its final results in 1977 ⁹, and it indeed allowed (when combined with the numerical studies of Roy equations) to constrain the value of the $I = 0$ S -wave scattering length to within a rather narrow range: $a_0^0 = 0.26 \pm 0.05$. Until today this

was the best experimental information available on this observable. Now we have already preliminary results¹⁰ from a new experiment that has been operating at Brookhaven, analysing more than $4 \cdot 10^5$ K_{e4} decays (to be compared to the $3 \cdot 10^4$ of the previous experiment⁹).

The prospects of a new generation of high-statistics experiment on K_{e4} decays and theoretical work on the calculation of the scattering lengths made necessary a revival of the numerical treatment of Roy equations. An extensive work has been made in¹¹, whose highlights I will now briefly summarize:

1. Given the strong dominance of the S - and P -waves at low energy, Roy equations have been solved only for these, and only on the interval $4M_\pi^2 < s < s_0 = (0.8 \text{ GeV})^2$, the lower half of their range of validity (which has been rigorously proven to extend up to $\sqrt{s_1} = 1.15 \text{ GeV}$). In that region the contributions generated by inelastic channels can be safely neglected. In the interval from s_0 to $s_2 = (2 \text{ GeV})^2$, the imaginary parts have been evaluated with the available experimental information, whereas above s_2 , a theoretical representation, based on Regge asymptotics has been used.
2. Unitarity converts the Roy equations for the S - and P -waves into a set of three coupled integral equations for the corresponding phase shifts: The real part of the partial wave amplitudes is given by a sum of known contributions (subtraction polynomial, integrals over the region $s_0 < s < s_2$ and driving terms) and certain integrals over their imaginary parts, extending from threshold to s_0 . Since unitarity relates the real and imaginary parts in a nonlinear manner, these equations are inherently nonlinear and cannot be solved explicitly.
3. Several mathematical properties of such integral equations are known: In particular, the existence and uniqueness of the solution is guaranteed only if the matching point s_0 is taken in the region between the place where the P -wave phase shift goes through 90° and the energy where the $I = 0$ S -wave does the same. As this range is quite narrow ($0.78 \text{ GeV} < \sqrt{s_0} < 0.86 \text{ GeV}$), there is little freedom in the choice of the matching point.
4. A second consequence of the mathematical structure of the Roy equations is that, for a given input and for a random choice of the two subtraction constants, the solution has a cusp at s_0 : In the vicinity of the matching point, the solution in general exhibits unphysical behaviour. The strength of the cusp is very sensitive to the value of a_0^2 . In fact, it was found that the cusp disappears if that value is tuned properly. Treating the imaginary parts as known, the requirement that the solution is free of cusps at the matching point determines the value of a_0^2 as a function of a_0^0 .
5. The input used for the imaginary parts above the matching point is subject to considerable uncertainties. In this framework, the values of the S - and P -

wave phase shifts at the matching point represent the essential parameters in this regard. The data on the pion form factor, obtained from the processes $e^+e^- \rightarrow \pi^+\pi^-$ and $\tau \rightarrow \pi^-\pi^0\nu_\tau$, very accurately determine the behaviour of the P -wave phase shift in the region of the ρ -resonance, thus constraining the value of $\delta_1^1(s_0)$ to a remarkably narrow range. The phase shifts extracted from the reaction $\pi N \rightarrow \pi\pi N$ constrain rather strictly the difference $\delta_0^0 - \delta_1^1$, better than either of the two phases individually. Since the P -wave is known very accurately from the leptonic processes mentioned above, this implies that $\delta_0^0(s_0)$ is also known rather well. The experimental information concerning δ_0^2 , on the other hand, is comparatively meagre.

6. The uncertainties in the experimental input for the imaginary parts and those in the driving terms turn the universal curve into a band in the (a_0^0, a_0^2) plane. Outside this “universal band”, the Roy equations do not admit physically acceptable solutions that are consistent with what is known about the behaviour of the imaginary parts above the matching point.

7. The Olsson sum rule relates the combination $2a_0^0 - 5a_0^2$ of scattering lengths to an integral over the imaginary parts of the amplitude. Evaluating the integral, it was found that the sum rule is satisfied within a band that has a large overlap with the “universal band” mentioned above. It is by no means built in from the start that the two requirements can simultaneously be met – the fact that this is the case represents a rather thorough check of our analysis.

8. The admissible region can be constrained further if use is made of experimental data below the matching point. At the moment there are two main sources of information on $\pi\pi$ scattering below 0.8 GeV: A few data points for the $I = 2$ S -wave phase shift – which will, unfortunately, not be improved in the foreseeable future – and a few data points on $\delta_0^0 - \delta_1^1$ very close to threshold, from K_{e4} decays. The latter provide an important constraint.

9. The Roy equation analysis is the only method that allows one to reliably translate low-energy data on the scattering amplitude into values for the scattering lengths. As discussed above, the available data do correlate the value of a_0^2 with the one of a_0^0 . Unfortunately, however, the value of a_0^0 is not strongly constrained: In agreement with earlier analyses, it was found that these data are consistent with any value of a_0^0 in the range from 0.18 to 0.3.

10. The two subtraction constants a_0^0, a_0^2 are the essential parameters at low energies: If these were known, the method would allow one to calculate the S - and P -wave phase shifts below 0.8 GeV to an amazing degree of accuracy. In particular one can evaluate the amplitude also at unphysical points. This is very important if one wants to use at best the information contained in the chiral representation of the $\pi\pi$ scattering amplitude, as we will see in the following section.

3 Matching the chiral and the dispersive representation

The Weinberg formula for the S -wave, $I = 0$ scattering lengths (1) is subject to substantial corrections. The physical reason for the size of these corrections is the strong rescattering of pions in the $I = 0$ channel. The leading order formula fails to fully describe such effects because unitarity corrections only show up at next-to-leading order and beyond. On the other hand, if one uses the Weinberg amplitude below threshold and away from it, these unitarity effects will become less and less important, and the nominal parameter of the chiral expansion (M_π^2 in 1 GeV^2 units, a number of the order of a few percent) will start to dictate the actual size of the corrections.

The fact that in the Roy equations the S -wave scattering lengths appear as the subtraction constants is only a matter of choice: one could as well subtract at an unphysical point, without changing any of the physical results. A combination of the two approaches that exploits the strengths of both is therefore possible: the chiral expansion may provide accurate information on the two subtraction constants, by fixing the value of the amplitude way below threshold, but would be less precise in the evolution of the amplitude up to threshold and above – there the dispersive framework based on the solution of Roy equations is much better. This program has been carried out in Ref. ⁶, making full use of the information contained in the two-loop chiral representation of the amplitude. In the following I will briefly describe some of the details.

3.1 Chiral representation of the scattering amplitude

The two-loop representation yields the first three terms in the low energy expansion of the partial waves:

$$t_\ell^I(s) = t_\ell^I(s)_2 + t_\ell^I(s)_4 + t_\ell^I(s)_6 + O(p^8) . \quad (5)$$

Since inelastic reactions start showing up only at $O(p^8)$, unitarity implies

$$\text{Im } t_\ell^I(s) = \sigma(s) |t_\ell^I(s)|^2 + O(p^8) , \quad \sigma(s) = (1 - 4M_\pi^2/s)^{\frac{1}{2}} . \quad (6)$$

The condition fixes the imaginary parts of the two-loop amplitude in terms of the one-loop representation. At leading order, the scattering amplitude is linear in the Mandelstam variables, so that only the S- and P-waves are different from zero. Unitarity therefore implies that, up to and including $O(p^6)$, only these partial waves develop an imaginary part. Accordingly, the chiral representation of the scattering amplitude can be written as ¹⁴

$$A(s, t, u) = C(s, t, u) + 32\pi \left\{ \frac{1}{3} U^0(s) + \frac{3}{2} (s - u) U^1(t) + \frac{3}{2} (s - t) U^1(u) \right.$$

$$+\frac{1}{2}U^2(t)+\frac{1}{2}U^2(u)-\frac{1}{3}U^2(s)\}+O(p^8) \ , \quad (7)$$

where $C(s, t, u)$ is a crossing symmetric polynomial,

$$C(s, t, u) = c_1 + s c_2 + s^2 c_3 + (t - u)^2 c_4 + s^3 c_5 + s (t - u)^2 c_6 \ . \quad (8)$$

The functions $U^0(s)$, $U^1(s)$ and $U^2(s)$ describe the “unitarity corrections” associated with s -channel isospin $I = 0, 1, 2$, respectively. In view of $\text{Im } t_\ell^I(s)_6 \propto s^3$, several subtractions are needed for the dispersive representation of these functions to converge. We subtract at $s = 0$ and set

$$U^I(s) = \frac{s^{4-\epsilon_I}}{\pi} \int_{4M_\pi^2}^{\infty} ds' \frac{\sigma(s') t^I(s')_2 \{t^I(s')_2 + 2 \text{Re } t^I(s')_4\}}{s'^{4-\epsilon_I} (s' - 4M_\pi^2)^{\epsilon_I} (s' - s)} \ . \quad (9)$$

The subtraction constants are collected in $C(s, t, u)$. As only the S - and P -waves enter, we have dropped the lower index, $\{t^0, t^1, t^2\} = \{t_0^0, t_1^1, t_0^2\}$. For kinematic reasons, the integrand of the P -wave differs from the one of the S -waves: $\{\epsilon_0, \epsilon_1, \epsilon_2\} = \{0, 1, 0\}$. It is straightforward to check that the result of the two-loop calculation³ is indeed of this structure.

The two-loop result for the amplitude specifies how the coefficients c_1, \dots, c_6 depend on the quark masses³ (for explicit expressions, see¹⁵). These formulae, in particular contain Weinberg’s low energy theorems, which in this language state that the leading terms in the expansion of the first two coefficients are fixed by M_π and F_π : $c_1 = -M_\pi^2/F_\pi^2 + \dots$, $c_2 = 1/F_\pi^2 + \dots$. At first order, the constants $\ell_1, \ell_2, \ell_3, \ell_4$ from \mathcal{L}_4 ¹⁶ enter, and at second order, the chiral representation of the scattering amplitude involves the couplings r_1, \dots, r_6 from \mathcal{L}_6 ¹⁷. It is useful to distinguish two categories of coupling constants:

- a. Terms that survive in the chiral limit.* Four of the coupling constants that enter the two-loop representation of the scattering amplitude belong to this category: ℓ_1, ℓ_2, r_5, r_6 .
- b. Symmetry breaking terms.* The corresponding vertices are proportional to a power of the quark mass and involve the coupling constants $\ell_3, \ell_4, r_1, r_2, r_3, r_4$.

The constants of the first category show up in the momentum dependence of the scattering amplitude, so that these couplings may be determined phenomenologically. The symmetry breaking terms, on the other hand, specify the dependence of the amplitude on the quark masses. Since these cannot be varied experimentally, information concerning the second category of coupling constants can only be obtained from sources other than $\pi\pi$ scattering. The constants r_n from \mathcal{L}_6 only generate tiny effects, so that crude theoretical

estimates suffice, but the couplings ℓ_3 and ℓ_4 from \mathcal{L}_4 do play an important role.

The crucial parameter that distinguishes the standard framework from the one proposed in Ref. ¹⁸ is ℓ_3 . This coupling constant determines the first order correction in the Gell-Mann-Oakes-Renner-relation: $M_\pi^2 = 2Bm \{1 - \frac{1}{2}\xi\bar{\ell}_3 + O(\xi^2)\}$, $\xi = 2Bm/(16\pi^2 F_\pi^2)$. The value of ℓ_3 is not known accurately. Numerically, however, a significant change in the prediction for the scattering lengths can only arise if the crude estimate in Ref. ¹⁶,

$$\bar{\ell}_3 = 2.9 \pm 2.4, \quad (10)$$

should turn out to be entirely wrong. We do not make an attempt at reducing the uncertainty in ℓ_3 within the standard framework, but will explicitly indicate the sensitivity to this coupling constant.

Chiral symmetry implies that the coupling constant ℓ_4 also shows up in the expansion of the scalar radius in powers of the quark masses ²:

$$\langle r^2 \rangle_s = \frac{3}{8\pi^2 F_\pi^2} \left\{ \bar{\ell}_4 - \frac{13}{12} + \xi \Delta_r + O(\xi^2) \right\}. \quad (11)$$

As pointed out in Ref. ¹⁹, the scalar radius can be determined on the basis of a dispersive evaluation of the scalar form factor. The result ⁶

$$\langle r^2 \rangle_s = 0.61 \pm 0.04 \text{ fm}^2, \quad (12)$$

is an update of the value given in Ref. ¹⁹ and is consistent with earlier estimates of the low energy constant ℓ_4 based on the symmetry breaking seen in F_K/F_π or on the decay $K \rightarrow \pi \ell \nu$ ²⁰, but is considerably more accurate. Since the chiral representation of the scalar form factor is known to two loops ²¹, the dependence of the correction Δ_r on the quark masses is also known. In addition to ℓ_1, \dots, ℓ_4 , the explicit expression involves a further term, r_{S2} , from \mathcal{L}_6 ¹⁷. In the following, we use this representation to eliminate the parameter ℓ_4 in favour of the scalar radius.

These numerical estimates of these low-energy constants are important when analyzing the corrections to low-energy theorems. Let us look, e.g., at the following two relations among the coefficients c_1, \dots, c_4 which are dictated by chiral symmetry. Consider the combinations

$$C_1 \equiv F_\pi^2 \{c_2 + 4M_\pi^2(c_3 - c_4)\}, \quad C_2 \equiv \frac{F_\pi^2}{M_\pi^2} \{-c_1 + 4M_\pi^4(c_3 - c_4)\}. \quad (13)$$

Chiral symmetry implies that, if the quark masses are turned off, both C_1 and C_2 tend to 1. The contributions from c_3 and c_4 ensure that the first order corrections only involve the symmetry breaking couplings ℓ_3 and ℓ_4 .

Eliminating ℓ_4 in favour of the scalar radius, the low energy theorems take the form

$$\begin{aligned} C_1 &= 1 + \frac{M_\pi^2}{3} \langle r^2 \rangle_s + \frac{23\xi}{420} + \xi^2 \Delta_1 + O(\xi^3) , \\ C_2 &= 1 + \frac{M_\pi^2}{3} \langle r^2 \rangle_s + \frac{\xi}{2} \left\{ \bar{\ell}_3 - \frac{17}{21} \right\} + \xi^2 \Delta_2 + O(\xi^3) . \end{aligned} \quad (14)$$

At first nonleading order, C_1 is fully determined by the contribution from the scalar radius, while C_2 also contains a contribution from ℓ_3 . Inserting the values $\langle r^2 \rangle_s = 0.61 \text{ fm}^2$ and $\bar{\ell}_3 = 2.9$ and ignoring the two-loop corrections Δ_1, Δ_2 , we obtain $C_1 = 1.103$, $C_2 = 1.117$. The value of C_2 differs little from C_1 – as stated above, the estimate (10) implies that the contributions from ℓ_3 are very small.

3.2 Phenomenological representation of the scattering amplitude

If one neglects the contribution of the absorptive parts from D waves and higher, the $\pi\pi$ scattering amplitude can be written as a combination of functions of a single variable that only have a right-hand cut – exactly like in the two-loop chiral representation. The imaginary parts that appear inside the dispersive integrals will be evaluated from the solutions of the Roy equations – since these are based on phenomenological information we will refer to this as the phenomenological representation. We subtract the relevant dispersion integrals in the same manner as for the chiral representation:

$$\bar{W}^I(s) = \frac{s^{4-\epsilon_I}}{\pi} \int_{4M_\pi^2}^{\infty} ds' \frac{\text{Im } t^I(s')}{s'^{4-\epsilon_I}(s - 4M_\pi^2)^{\epsilon_I}(s' - s)} . \quad (15)$$

Since all other contributions can be replaced by a polynomial, the phenomenological amplitude takes the form

$$\begin{aligned} A(s, t, u) &= 16\pi a_0^2 + \frac{4\pi}{3M_\pi^2} (2a_0^0 - 5a_0^2) s + \bar{P}(s, t, u) \\ &+ 32\pi \left\{ \frac{1}{3} \bar{W}^0(s) + \frac{3}{2}(s-u) \bar{W}^1(t) + \frac{3}{2}(s-t) \bar{W}^1(u) \right. \\ &\quad \left. + \frac{1}{2} \bar{W}^2(t) + \frac{1}{2} \bar{W}^2(u) - \frac{1}{3} \bar{W}^2(s) \right\} + O(p^8) . \end{aligned} \quad (16)$$

We have explicitly displayed the contributions from the subtraction constants a_0^0 and a_0^2 . The term $\bar{P}(s, t, u)$ is a crossing symmetric polynomial

$$\bar{P}(s, t, u) = \bar{p}_1 + \bar{p}_2 s + \bar{p}_3 s^2 + \bar{p}_4 (t-u)^2 + \bar{p}_5 s^3 + \bar{p}_6 s(t-u)^2 . \quad (17)$$

Its coefficients can be expressed in terms of integrals over the imaginary parts of the partial waves. Explicit expressions can be found in Ref. ¹⁵. In the following, the essential point is that the coefficients $\bar{p}_1, \dots, \bar{p}_6$ can be determined phenomenologically.

3.3 Matching the two representations

In their common domain of validity, the two representations of the scattering amplitude specified above agree, provided the parameters occurring therein are properly matched:

$$C(s, t, u) = 16\pi a_0^2 + \frac{4\pi}{3M_\pi^2} (2a_0^0 - 5a_0^2) s + \bar{P}(s, t, u) + O(p^8) . \quad (18)$$

Since the main uncertainties in the coefficients of the polynomial $\bar{P}(s, t, u)$ arise from their sensitivity to the scattering lengths a_0^0, a_0^2 , the above relations essentially determine the coefficients c_1, \dots, c_6 in terms of these two observables. The same then also holds for the quantities C_1, C_2 defined in eq. (13). The corresponding low energy theorems for a_0^0 and a_0^2 are of the form

$$a_0^0 = \frac{7M_\pi^2 C_0}{32\pi F_\pi^2} + M_\pi^4 \alpha_0 + O(m^4), \quad a_0^2 = -\frac{M_\pi^2 C_2}{16\pi F_\pi^2} + M_\pi^4 \alpha_2 + O(m^4), \quad (19)$$

with $C_0 \equiv \frac{1}{7}(12C_1 - 5C_2)$. The terms α_0, α_2 stand for integrals over the imaginary parts of the partial waves that can be worked out from the available experimental information. Formula (19) clearly shows how the matching works: chiral symmetry provides accurate information on the unphysical quantities C_0 and C_2 (14): the relation between these quantities and the scattering lengths involves the terms α_0 and α_2 , which are best calculated from the explicit numerical solution of the Roy equations discussed in the previous section.

3.4 Results for a_0^0 and a_0^2 and other threshold parameters

Inserting the one-loop prediction for C_1, C_2 in the relations (19) and solving for a_0^0, a_0^2 , we obtain the following first order results:

$$a_0^0 = 0.2195, \quad a_0^2 = -0.0446, \quad 2a_0^2 - 5a_0^0 = 0.662. \quad (20)$$

The two-loop corrections Δ_1 and Δ_2 involve the coupling constants ℓ_1, ℓ_2, ℓ_3 , the scalar radius, as well as the terms r_1, \dots, r_4, r_{S2} from \mathcal{L}_6 . The size of the contributions from the latter may be estimated with the resonance model described in Refs. ^{3,21}. The constants ℓ_1, ℓ_2 can then be determined

	Roy	CHPT $O(p^6)$ Roy	CHPT $O(p^6)$ K_{e4}
a_0^0	$.220 \pm .005$.215	$.219 \pm .005$
$-10a_0^2$	$.444 \pm .001$.445	$.420 \pm .010$
$10b_0^0$	2.76 ± 0.06	2.68	2.79 ± 0.11
$-10^2b_0^2$	8.03 ± 0.12	8.08	7.56 ± 0.21
$10^2a_1^1$	3.79 ± 0.05	3.80	3.78 ± 0.21
$10^3b_1^1$	5.67 ± 0.13	5.37	5.9 ± 1.2
$10^3a_2^0$	1.75 ± 0.03	1.76	2.2 ± 0.4
$10^4a_2^2$	1.70 ± 0.13	1.72	2.9 ± 1

Table 1. Comparison of the numerical values of threshold parameters as obtained from the Roy solutions (after matching with CHPT), or CHPT. In the latter case the low-energy constants can be fixed either by the matching with Roy equations (third column), or by a phenomenological analysis of K_{e4} form factors at two loops ²² (fourth column).

numerically with the phenomenological values of c_3 and c_4 . The resulting two-loop corrections for the scattering lengths are very small. The numerical result is sensitive to the value of the scale μ at which the renormalized coupling constants $r_n^r(\mu)$ are assumed to be saturated by the resonance contributions. In the following, we use the resonance model at the scale $\mu = M_\rho$ and take the range $500\text{MeV} \leq \mu \leq 1\text{GeV}$ as an estimate for the uncertainties to be attached to the two-loop corrections.

A careful evaluation of the uncertainties coming from the phenomenological input leads to

$$\begin{aligned} a_0^0 &= 0.220 \pm 0.001 - 0.0017 \Delta\ell_3 + 0.027 \Delta_{r^2}, \\ a_0^2 &= -0.0444 \pm 0.0003 - 0.0004 \Delta\ell_3 - 0.004 \Delta_{r^2}, \end{aligned} \quad (21)$$

with $\bar{\ell}_3 = 2.9 + \Delta\ell_3$, $\langle r^2 \rangle_s = 0.61 \text{ fm}^2 (1 + \Delta_{r^2})$. Inserting the estimates (10), (12), we arrive at our final result:

$$\begin{aligned} a_0^0 &= 0.220 \pm 0.005, & a_0^2 &= -0.0444 \pm 0.0010, \\ 2a_0^0 - 5a_0^2 &= 0.663 \pm 0.006, & a_0^0 - a_0^2 &= 0.265 \pm 0.004. \end{aligned} \quad (22)$$

Having fixed the subtraction constants of the Roy equations by matching to the chiral representation we can now use the Roy solutions to evaluate the $\pi\pi$ scattering amplitude at any energy. As an example we display in table 1 the results for other threshold parameters. For comparison we also show the numerical values obtained from a direct evaluation of the chiral two-loop amplitude. In this case we have to specify how we fix the low-energy constants: since the matching procedure does provide values for the latter, this is the most obvious choice. On the other hand, in Ref. ²² a two-loop analysis of the

K_{e4} form factors has lead to an alternative determination of the low-energy constants. In table 1 we give both sets of values for the threshold parameters. The comparison shows clearly that two-loop chiral perturbation theory works very well in describing both $\pi\pi$ scattering and K_{e4} decays. On the other hand, the smallest error bars for the threshold parameters are obtained by combining the chiral representation and Roy equations.

4 Conclusions

We have reviewed the current status of the theory of $\pi\pi$ scattering. The combination of numerical solutions of Roy equations and two-loop chiral perturbation theory has lead to a remarkably precise description of this reaction at low energy. As an example we have discussed the numerical values of the two S -wave scattering lengths, which are given with an uncertainty of a few percent, and of other threshold parameters, with somewhat larger error bars.

The new experiments at Brookhaven ¹⁰ and at DAΦNE ¹² on K_{e4} decays will yield more precise information in the very near future. On the basis of the preliminary data of the E865 collaboration we have estimated in Ref. ⁶ the present allowed range for a_0^0 to be between 0.20 and 0.25, in perfect agreement with our prediction (22). Moreover, the pionic atom experiment under way at CERN ¹³ will allow a direct measurement of $|a_0^0 - a_0^2|$. These experimental tests will tell us whether we have reached a full, accurate understanding of the low-energy dynamics of pion interactions.

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